Episode 17

Describing Motion of Rigid Bodies - Part 2: Applications of Rigid Body Kinematics

ENGN0040: Dynamics and Vibrations

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Topics for todays class

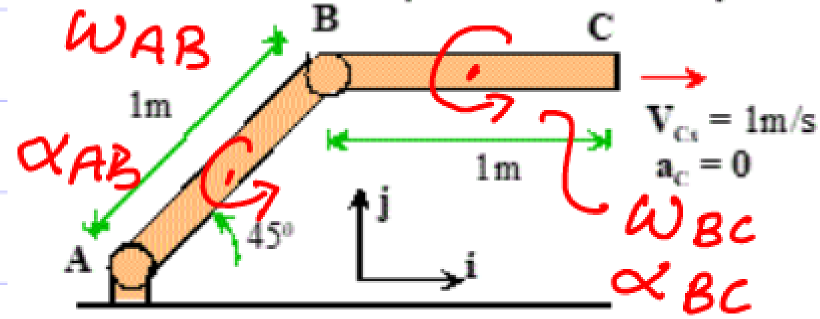
Applications of rigid body kinematics

1. Analyzing motion in connected systems of rigid bodies
2. Gears
3. Pulleys
4. The rolling wheel

6.2 Analyzing motion in systems of rigid bodies

General Problem Given $\underline{v}$, $\underline{a}$ of two points in system find $\underline{\omega}$, $\underline{\alpha}$ for each member

Problem: how to rotate joints at A, B to move end C with prescribed velocity



Procedure:

(1) Introduce unknown $\underline{\omega}$, $\underline{\alpha}$ for each rigid body

(2) Relate velocities of known points (A, C) using rigid body formulas and constraint equations at joints

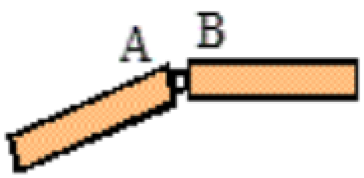
(3) Solve eq (2) for unknown $\underline{\omega}$

(4) Repeat (2) for accelerations

(5) Solve for unknown $\underline{\alpha}$

Example Constraints

Pin Joint

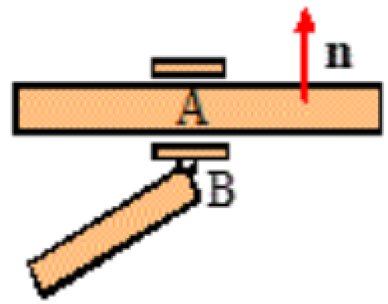


$\underline{V}_B = \underline{V}_A$

$\underline{a}_B = \underline{a}_A$

@ connected point

Slider Joint



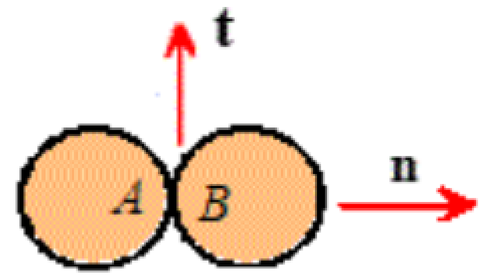
$\underline{V}_A \cdot \underline{n} = \underline{V}_B \cdot \underline{n}$

$\underline{a}_A \cdot \underline{n} = \underline{a}_B \cdot \underline{n}$

(or $V_{An} = V_{Bn}$

$a_{An} = a_{Bn}$)

Contact

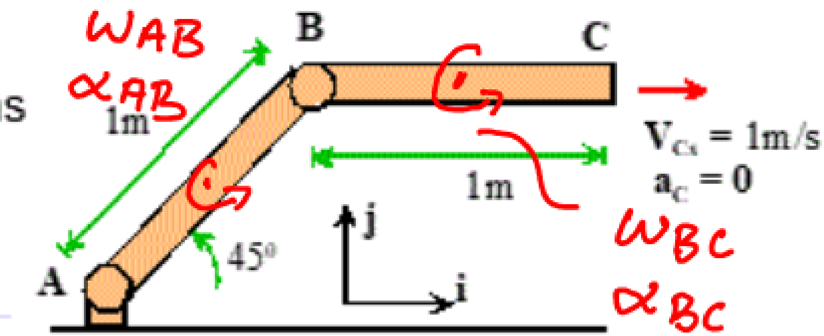


No slip

$\underline{V}_B = \underline{V}_A$

$\underline{a}_B \cdot \underline{t} = \underline{a}_A \cdot \underline{t}$

6.2.1 Example: Find the angular velocities and accelerations of the two actuators at A and B that will move point C with instantaneous velocity and acceleration $\mathbf{v}_C = 1m/s \quad \mathbf{a}_C = 0$



Velocities

$$\underline{V}_B - \underline{V}_A = \omega_{AB}\underline{k} \times (\underline{r}_B - \underline{r}_A) = \omega_{AB}\underline{k} \times (1/\sqrt{2}\,\underline{i} + 1/\sqrt{2}\,\underline{j})$$
$$= (\omega_{AB}/\sqrt{2})(-\underline{i} + \underline{j})$$

$$\underline{V}_C - \underline{V}_B = \omega_{BC}\underline{k} \times \underline{i} = \omega_{BC}\underline{j}$$

$$(1)+(2) \Rightarrow -(\omega_{AB}/\sqrt{2})\underline{i} + (\omega_{BC} + \omega_{AB}/\sqrt{2})\underline{j} = \underline{V}_C - \underline{V}_A$$

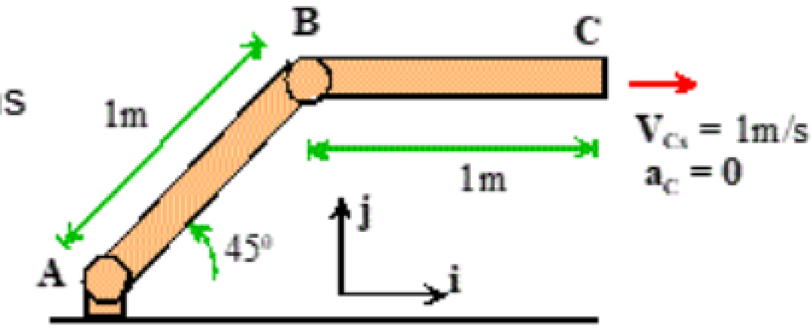
where $\underline{V}_C = 1\underline{i}$ and $\underline{V}_A = 0$.

$$\underline{i} \Rightarrow -\omega_{AB}/\sqrt{2} = 1 \Rightarrow \omega_{AB} = -\sqrt{2}\ \text{rad/s}$$
$$\underline{j} \Rightarrow \omega_{BC} + \omega_{AB}/\sqrt{2} = 0 \Rightarrow \omega_{BC} = 1\ \text{rad/s}$$

Actuator rates:

$$\omega_A = \omega_{AB} = -\sqrt{2}\ \text{rad/s}$$
$$\omega_B = \omega_{BC} - \omega_{AB} = (1+\sqrt{2})\ \text{rad/s}$$

6.2.1 Example: Find the angular velocities and accelerations of the two actuators at A and B that will move point C with instantaneous velocity and acceleration $\mathbf{v}_C = 1m/s$ $\mathbf{a}_C = 0$



Accelerations

$$\underline{a}_B - \underline{a}_A = \alpha_{AB}\underline{k} \times (\underline{r}_B - \underline{r}_A) - \omega_{AB}^2 (\underline{r}_B - \underline{r}_A) \qquad \omega_{AB} = -\sqrt{2}$$

$$= (\alpha_{AB}/\sqrt{2})(-\underline{i} + \underline{j}) - 2(\underline{i} + \underline{j})/\sqrt{2}$$

$$= -(\alpha_{AB} + 2)\underline{i}/\sqrt{2} + (\alpha_{AB} - 2)\underline{j}/\sqrt{2}$$

$$\underline{a}_c - \underline{a}_B = \alpha_{BC}\underline{k} \times \underline{i} - \omega_{BC}^2 \underline{i} \qquad \omega_{BC} = 1$$

$$= -\underline{i} + \alpha_{BC}\underline{j}$$

Add:

$$\underline{a}_c - \underline{a}_A = \underline{0} = -(\alpha_{AB}/\sqrt{2} + \sqrt{2} + 1)\underline{i} + (\alpha_{AB}/\sqrt{2} + \alpha_{BC} - \sqrt{2})\underline{j}$$

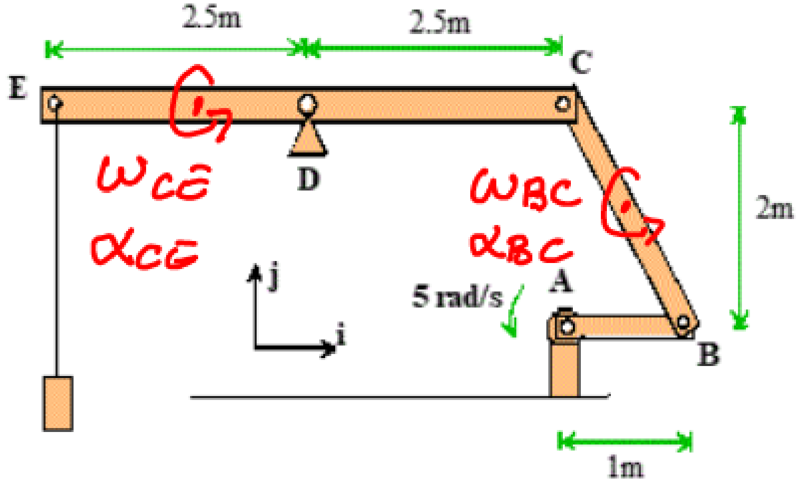
$\underline{i} \Rightarrow \quad \alpha_{AB} = -2 - \sqrt{2} \ rad/s^2$

$\underline{j} \Rightarrow \quad \alpha_{BC} = 1 + 2\sqrt{2} \ rad/s^2$

Actuators

$$\alpha_A = \alpha_{AB} = -(2 + \sqrt{2}) \ rad/s^2$$

$$\alpha_B = \alpha_{BC} - \alpha_{AB} = 3(1 + \sqrt{2}) \ rad/s^2$$



Note $\underline{v}_A = \underline{v}_D = \underline{0}$

$\underline{a}_A = \underline{a}_D = \underline{0}$

6.2.2 Example: Link AB rotates counterclockwise with constant angular speed 5 rad/s. Find the velocity and acceleration of point E.

Velocities

$$\underline{v}_B - \underline{v}_A = \omega_{AB}\underline{k} \times (\underline{r}_B - \underline{r}_A) = 5\underline{k} \times (1\underline{i}) = 5\underline{j}$$

$$\underline{v}_C - \underline{v}_B = \omega_{BC}\underline{k} \times (-\underline{i} + 2\underline{j}) = -2\omega_{BC}\underline{i} - \omega_{BC}\underline{j}$$

$$\underline{v}_D - \underline{v}_C = \omega_{CE}\underline{k} \times (-2.5\underline{i}) = -2.5\omega_{CE}\underline{j}$$

Add:

$$\underline{v}_D - \underline{v}_A = \underline{0} = -2\omega_{BC}\underline{i} + (5 - \omega_{BC} - 2.5\omega_{CE})\underline{j}$$

$\underline{i} \Rightarrow$ $\omega_{BC} = 0$ $\quad \underline{j} \Rightarrow$ $\omega_{CE} = 2$ rad/s

Finally $\underline{v}_E - \underline{v}_D = \omega_{CE}\underline{k} \times (-2.5\underline{i}) = -2.5\omega_{CE}\underline{j}$

$\Rightarrow$ $\underline{v}_E = -5\underline{j}$ m/s

Accelerations

$$\underline{a}_B - \underline{a}_A = \alpha_{AB}\underline{k} \times (\underline{r}_B - \underline{r}_A) - \omega_{AB}^2(\underline{r}_B - \underline{r}_A) = -25\underline{i}$$

$$\underline{a}_C - \underline{a}_B = \alpha_{BC}\underline{k} \times (-\underline{i} + 2\underline{j}) - 0^2(-\underline{i} + 2\underline{j}) = -2\alpha_{BC}\underline{i} - \alpha_{BC}\underline{j}$$

$$\underline{a}_D - \underline{a}_C = \alpha_{CE}\underline{k} \times (-2.5\underline{i}) - 4(-2.5\underline{i}) = 10\underline{i} - 2.5\alpha_{CE}\underline{j}$$

Add:

$$\underline{a}_D - \underline{a}_A = (-25 - 2\alpha_{BC} + 10)\underline{i} - (\alpha_{BC} + 2.5\alpha_{CE})\underline{j}$$

$\underline{i} \Rightarrow \quad \alpha_{BC} = -15/2 \text{ rad/s}^2 \qquad \underline{j} \Rightarrow \quad \alpha_{CE} = 3 \text{ rad/s}^2$

Finally $\quad \underline{a}_E - \underline{a}_D = \alpha_{CE}\underline{k} \times (-2.5\underline{i}) - 4(-2.5\underline{i})$

$$\Rightarrow \quad \boxed{\underline{a}_E = (10\underline{i} - 7.5\underline{j}) \text{ m/s}^2}$$

6.3 Gears, Pulleys and the Rolling Wheel

6.3.1 Simple Gear Pair

Gears can be viewed as two touching cylinders

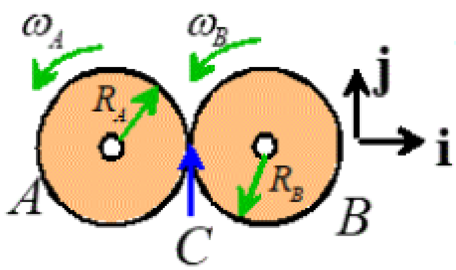
Velocities of A, B equal at C

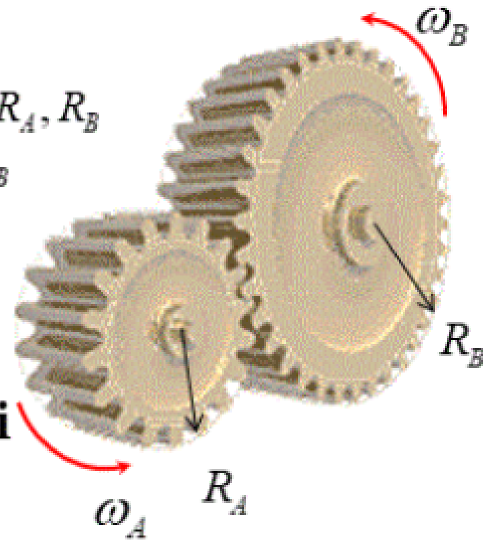
Given one of:

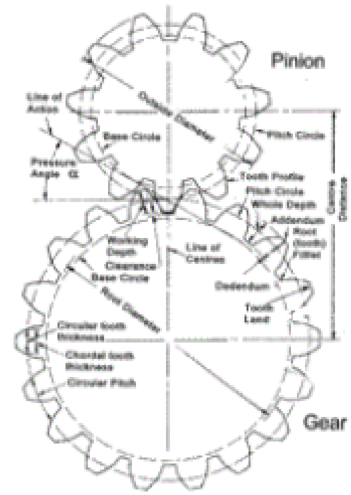
(a) 'Pitch Circle Radii' R_A, R_B

(b) Nos. of teeth N_A, N_B

Find ω_B / ω_A







Rigid body formula

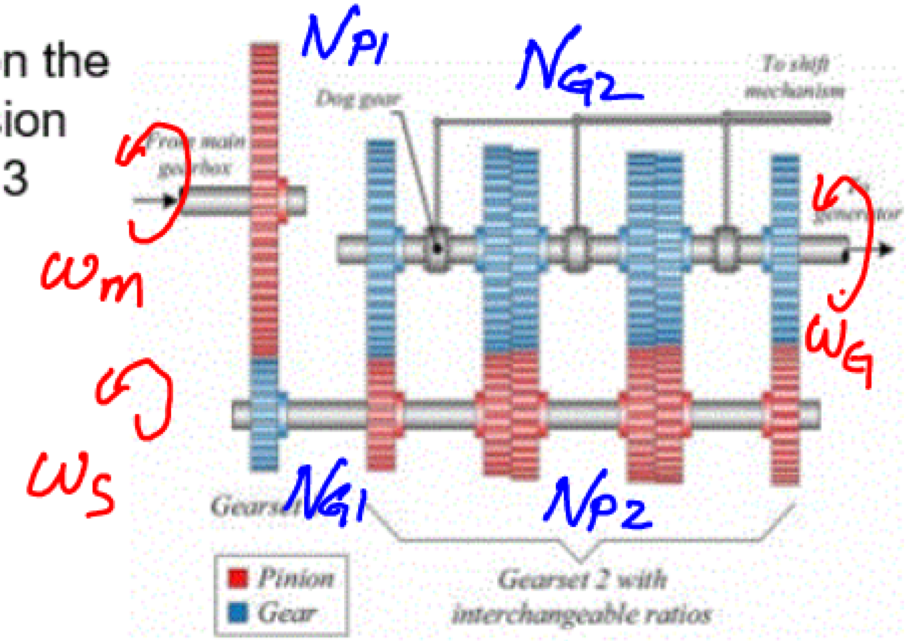
A: $\underline{V}_C - \underline{O} = \omega_A \underline{k} \times (R_A \underline{i}) = R_A \omega_A \underline{j}$

B: $\underline{V}_C - \underline{O} = \omega_B \underline{k} \times (-R_B \underline{i}) = -R_B \omega_B \underline{j}$

$\Rightarrow \omega_A R_A = -\omega_B R_B \Rightarrow$ $\boxed{\dfrac{\omega_B}{\omega_A} = -\dfrac{R_A}{R_B} = -\dfrac{N_A}{N_B}}$

Spacing between teeth equal $\Rightarrow \dfrac{2\pi R_A}{N_A} = \dfrac{2\pi R_B}{N_B} \Rightarrow \dfrac{R_A}{R_B} = \dfrac{N_A}{N_B}$

6.3.2 Example: The table lists the numbers of teeth on the seven gear/pinion pairs for the wind turbine transmission shown. Calculate the ratio $\omega_{Generator} / \omega_{main}$ with Gear 3 connected to the output shaft.



		Gearset 2					
	Gearset 1	Gear 1	Gear 2	Gear 3	Gear 4	Gear 5	Gear 6
Pinion teeth	135	49	53	56	58	59	60
Gear teeth	50	101	97	94	92	91	90

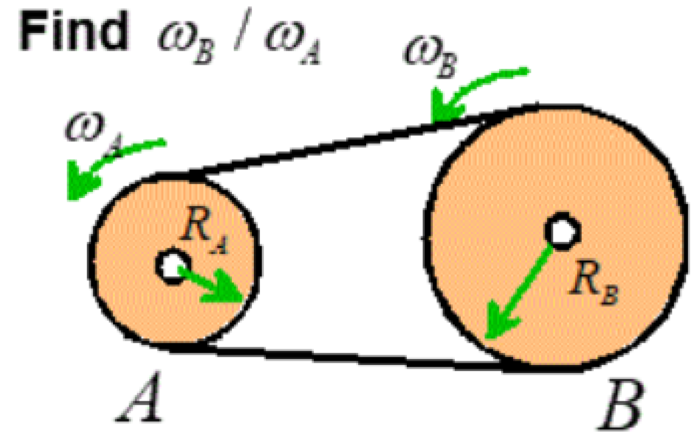
Gear formulas

$$\frac{\omega_s}{\omega_m} = -\frac{N_{P1}}{N_{G1}} \qquad \frac{\omega_G}{\omega_s} = -\frac{N_{P2}}{N_{G2}}$$

Hence

$$\frac{\omega_G}{\omega_m} = \frac{N_{P1}\,N_{P2}}{N_{G1}\,N_{G2}} = \frac{135}{50} \times \frac{56}{94} = \frac{378}{235}$$

6.3.3 Pulley Pairs



Notes:

(1) Points on pulley circumference are in circular motion

(2) Belt inextensible

$\Rightarrow$ A, B have same speed

Hence $\omega_A R_A = \omega_B R_B$

$$\Rightarrow \boxed{\frac{\omega_B}{\omega_A} = \frac{R_A}{R_B}}$$

6.3.4 Epicyclic Gears

Simple epicyclic gear problem

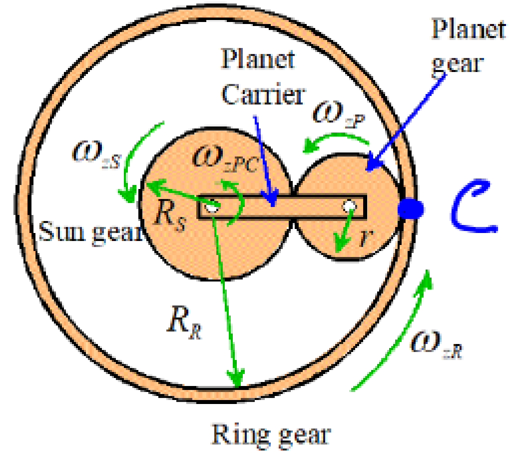
The planet carrier is stationary $\omega_{zPC} = 0$

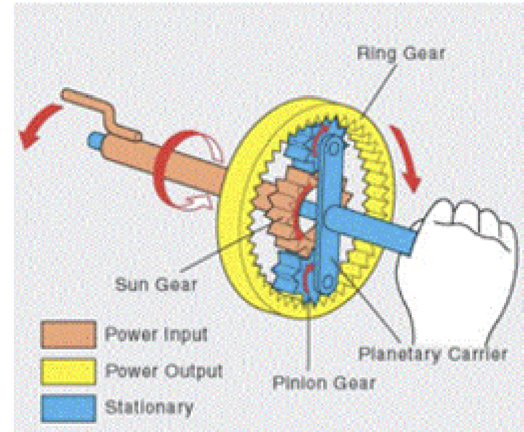
The sun gear has angular speed ω_{zS}

Find the angular speeds of the planet ω_{zP} and ring gear ω_{zR}

Given: Gear radii R_S, R_R

No. teeth N_S, N_R





Geometry: $R_R = R_S + 2r \Rightarrow$

$$r = (R_R - R_S)/2$$
$$N_P = (N_R - N_S)/2$$

Note sun & planet are a standard gear pair

$$\Rightarrow \frac{\omega_{zP}}{\omega_{zS}} = -\frac{R_S}{r}$$

Planet & ring have same speed @ C

$$\Rightarrow \omega_{zP} r = \omega_{zR} R_R \Rightarrow \frac{\omega_{zR}}{\omega_{zS}} = -\frac{R_S}{R_R} = -\frac{N_S}{N_R}$$

General epicyclic gear problem

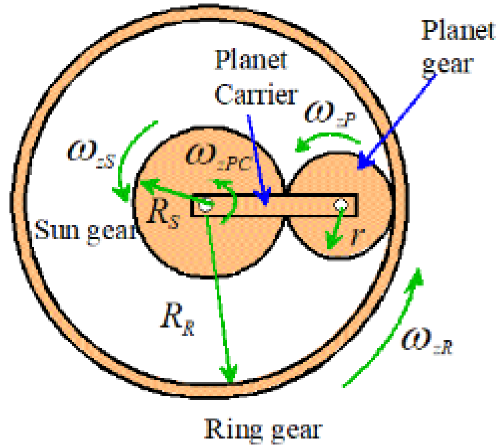
The planet carrier has angular speed ω_{zPC}

The sun gear has angular speed ω_{zS}

Find the angular speeds of the planet ω_{zP} and ring gear ω_{zR}

Given: Gear radii R_S, R_R

No. teeth N_S, N_R



Solution Procedure

Adopt ref frame rotating with planet carrier

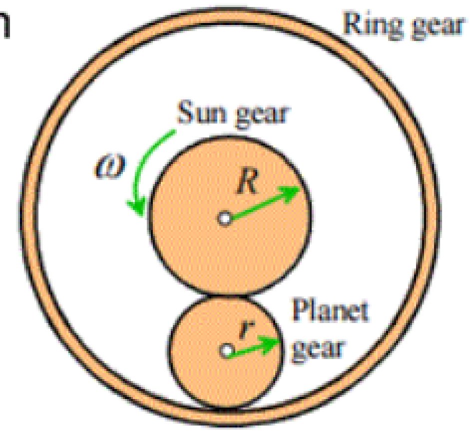
⇒ subtract ω_{zPC} from all angular speeds

⇒ Planet carrier stationary – use simple formulas

$$\frac{\omega_{zP}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}} = -\frac{R_S}{r} = \frac{-2R_S}{R_R-R_S} = \frac{-2N_S}{N_R-N_S}$$

$$\frac{\omega_{zR}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}} = -\frac{R_S}{R_R} = -\frac{N_S}{N_R}$$

6.3.5 Example The ring gear is stationary and the sun gear rotates counterclockwise with angular speed ω_S. What is the angular speed of the planet gear?



Use formulas

$$\frac{\omega_R - \omega_{PC}}{\omega_S - \omega_{PC}} = -\frac{R_S}{R_R} \qquad (\omega_R = 0)$$

$$\Rightarrow \quad \omega_{PC}\,(1 + R_S/R_R) = \omega_S R_S/R_R$$

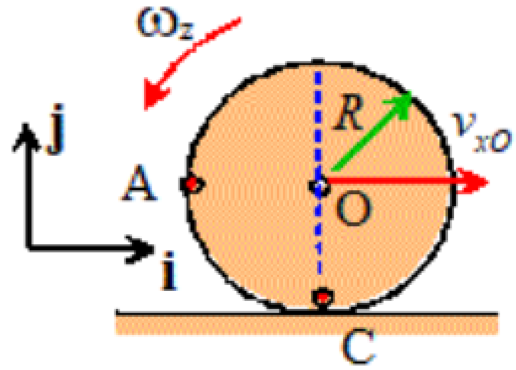
$$\Rightarrow \quad \omega_{PC} = \omega_S R_S / (R_R + R_S)$$

Geometry $\quad R_R = R_S + 2r \quad \Rightarrow \omega_{PC} = \omega_S R_S / 2(R_S + r)$

$$\frac{\omega_P - \omega_{PC}}{\omega_S - \omega_{PC}} = -\frac{R_S}{r} \quad \Rightarrow \quad \omega_P = \omega_{PC}\left(1 + \frac{R_S}{r}\right) - \omega_S \frac{R_S}{r}$$

$$\Rightarrow \quad \boxed{\omega_P = \frac{\omega_S R_S}{2r} - \frac{\omega_S R_S}{r} = -\frac{\omega_S R_S}{2r}}$$

6.3.6 The rolling wheel formulas



Find formulas relating $\underline{v}_o$, $\underline{a}_o$ to ω_z, α_z

Physics: Point C on wheel is stationary

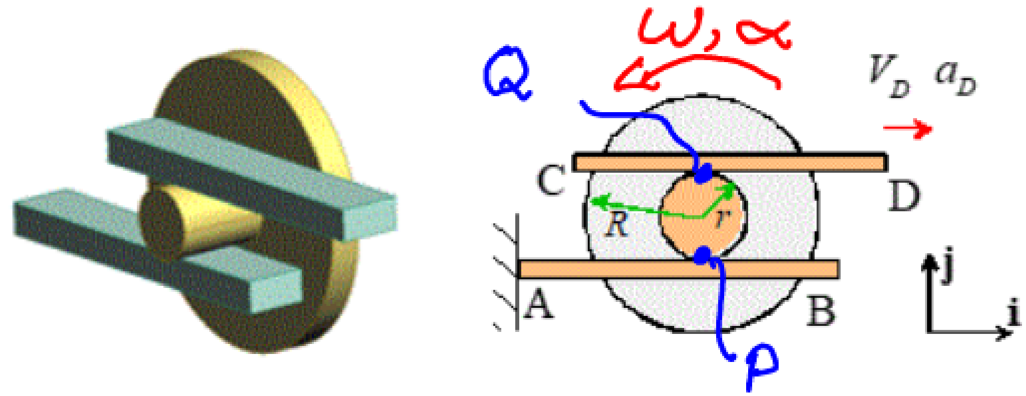
Rigid body formula: $\underline{v}_o - \underline{v}_c = \omega_z \underline{k} \times (R\underline{j})$

$$\Rightarrow \quad \boxed{\underline{v}_o = -\omega_z R \underline{i} \qquad v_{ox} = -\omega_z R}$$

Differentiate wrt time

$$\boxed{\underline{a}_o = -\alpha_z R \underline{i} \qquad a_{ox} = -\alpha_z R}$$

6.3.7 Example The figure shows a design for an 'inerter'. Point A is stationary and point D moves horizontally with speed V_D and acceleration a_D . Find formulas for the angular speed and acceleration of the flywheel.



The wheel/bars are rack-and-pinion gears (roll without slip)

Notes :

(1) All points on AB are stationary

(2) All points on CD have vel / accel $V_D \underline{i}$, $a_D \underline{i}$

(3) AB & pinion } have same velocity at { P
CD & pinion } { Q

Rigid body formula for pinion

$$\underline{V}_Q - \underline{V}_P = V_D \underline{i} = \omega \underline{k} \times (2r \underline{j}) = -2\omega r \underline{i}$$

$$\Rightarrow \quad \boxed{\omega = -V_D / (2r)}$$

Take time derivative

$$\boxed{\alpha = -a_D / (2r)}$$